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5-choosability of graphs with 2 crossings *

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Abstract: We show that every graph with two crossings is 5-choosable. We also prove that every graph which can be made planar by removing one edge is 5-choosable.

Key-words: list colouring, choosability, crossing number

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5-choisissabilité des graphes ayant deux croisements

Résumé : Nous montrons que tout graphe ayant deux croisements est 5-choisissable. Nous prouvons également que tout graphe qui peut être rendu planaire par la suppression d'une arête est 5-choisissable.

Mots-clés : coloration sur listes, choisissabilité, nombre de croisements

1 Introduction

The crossing number of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G in the plane.

The Four Colour Theorem states that, if a graph has crossing number zero (i.e. is planar), then it is 4-colourable. Deleting one vertex per crossing, it follows that $\chi(G) \leq 4 + cr(G)$. So it is natural to ask for the smallest integer $f(k)$ such that every graph G with crossing number at most k is $f(k)$-colourable? Settling a conjecture of Albertson [1], Schaefer [8] showed that $f(k) = O\left(k^{1/4}\right)$. This upper bound is tight up to a constant factor since $\chi(K_n) = n$ and $\mathrm{cr}(K_n) \leq \binom{|E(K_n)|}{2} = \binom{\binom{n}{2}}{2} \leq \frac{1}{8}n^4$.

The values of $f(k)$ are known for a number of small values of k. The Four Colour Theorem states $f(0) = 4$ and implies easily that $f(1) \leq 5$. Since $\mathrm{cr}(K_5) = 1$, we have $f(1) = 5$. Oporowski and Zhao [7] showed that $f(2) = 5$. Since $\mathrm{cr}(K_6) = 3$, we have $f(3) = 6$. Further, Albertson et al. [2] showed that $f(6) = 6$. Albertson then conjectured that if $\chi(G) = r$, then $\mathrm{cr}(G) \leq \mathrm{cr}(K_r)$. This conjecture was proved by Barát and Tóth [3] for $r \leq 16$.

A *list assignment* of a graph G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colours. An *L-colouring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-colouring, then it is *L-colourable*. A graph G is *k-choosable* if it is L-colourable for every list assignment L such that $|L(v)| \geq k$ for all $v \in V(G)$. The *choose number* of G, denoted by $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

Similarly to the chromatic number, one may seek for bounds on the choose number of a graph with few crossings or with independent crossings. Thomassen's Five Colour Theorem [10] states that if a graph has crossing number zero (i.e. is planar) then it is 5-choosable. A natural question is to ask whether the chromatic number is bounded in terms of its crossing number. Erman et al. [5] observed that Thomassen's result can be extended to graphs with crossing number at most 1. Deleting one vertex per crossing yields $\mathrm{ch}(G) \leq 4 + cr(G)$. Hence, what is the smallest integer $g(k)$ such that every graph G with crossing number at most k is $g(k)$-choosable? Obviously, since $\chi(G) \leq \mathrm{ch}(G)$, we have $f(k) \leq g(k)$.

In this paper, we extend Erman et al. result in two ways. We first show that every graph which can be made planar by the removal of an edge is 5-choosable (Theorem 3). We then prove that is $g(2) = 5$. In other words, every graph with crossing number 2 is 5-choosable[1]. This generalizes the result of Oporowski and Zhao [7] to list colouring.

2 Planar graphs plus an edge

In order to prove its Five Colour Theorem, Thomassen [10] showed a stronger result.

Definition 1. An *inner triangulation* is a plane graph such that every face of G is bounded by a triangle except its outer face which is bounded by a cycle.

Let G be a plane graph and x and y two consecutive vertices on its outer face F. A list assignment L of G is $\{x,y\}$*-suitable* if

- $|L(x)| \geq 1$, $|L(y)| \geq 2$,

[1] While writing this paper, we discovered that Dvořák et al. [4] independently proved this result. Their proof has some similarity to ours but is different. They prove by induction a stronger result, while we use the existence of a shortest path between the two crossings which satisfies some given properties.

- for every $v \in V(F) \setminus \{x,y\}$, $|L(v)| \geq 3$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

A list assignment of G is *suitable* if it is $\{x,y\}$-suitable for some vertices x and y on the outer face of G.

The following theorem is a straightforward generalization of Thomassen's five colour Theorem which holds for non-separable plane graphs.

Theorem 2 (Thomassen [10])**.** *If L is a suitable list assignment of a plane graph G then G is L-colourable.*

This result is the cornerstone of the following proof.

Theorem 3. *Let G be a graph. If G has an edge such that $G \setminus e$ is planar then* $\mathrm{ch}(G) \leq 5$.

Proof. Let $e = uv$ be an edge of G such that $G \setminus e$ is planar. Let G' be a planar triangulation containing $G \setminus e$ as a subgraph. Without loss of generality, we may assume that u is on the outer triangle of G'. The graph $G' - u$ has an outer cycle C' whose vertices are the neighbours of u in G'.

Let L be a 5-list assignment of G. Let $\alpha, \beta \in L(u)$. Let L' be the list-assignment of $G' - u$ defined by $L'(w) = L(w) \setminus \{\alpha, \beta\}$ if $w \in V(C')$ and $L'(w) = L(w)$ otherwise. Then L' is suitable. So $G' - u$ admits an L'-colouring by Theorem 2. This colouring may be extended into an L-colouring of G by assigning to u a colour in $\{\alpha, \beta\}$ different from the colour of v.

Hence G is 5-choosable. □

3 Graphs with two crossings

3.1 Preliminaries

We first recall the celebrated characterization of planar graphs due to Kuratowski [6]. See also [9] for a nice proof.

Theorem 4 (Kuratowski [6])**.** *A graph is planar if and only if it contains no minor isomorphic to either K_5 or $K_{3,3}$.*

Let G be a plane graph and x, y and z three distinct vertices on the outer face F of G. A list assignment L of G is (x,y,z)*-correct* if

- $|L(x)| = 1 = |L(y)|$ and $L(x) \neq L(y)$,
- $|L(z)| \geq 3$,
- for every $v \in V(F) \setminus \{x,y,z\}$, $|L(v)| \geq 4$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

If L is (x,y,z)-correct and $L(z) \geq 4$, we say that L is $\{x,y\}$*-correct*.

Lemma 5. *Let G be an inner triangulation and x and y two distinct vertices on the outer face of G. If L is an (x,y,z)-correct list assignment of G then G is L-colourable.*

Proof. We prove the result by induction on the number of vertices, the result holding trivially when $|V(G)| = 3$.

Suppose first that F has a chord xt. Then xt lies in two unique cycles in $F \cup xt$, one C_1 containing y and the other C_2. For $i = 1,2$, let G_i denote the subgraph induced by the vertices lying on C_i or inside it. By the induction hypothesis, there exists an L-colouring ϕ_1 of G_1. Let L_2 be the list assignment on G_2 defined by $L_2(t) = \{\phi_1(t)\}$ and $L_2(u) = L(u)$ if $u \in V(G_2) \setminus \{t\}$. Let $z' = z$ if $z \in V(C_2)$ and z' be any vertex of $V(C_2) \setminus \{x,t\}$ otherwise. Then L_2 is (x,t,z')-correct for G_2 so G_2 admits an L_2-colouring ϕ_2 by induction hypothesis. The union of ϕ_1 and ϕ_2 is an L-colouring of G.

Suppose now that x has exactly two neighbours u and v on F. Let $u, u_1, u_2 \dots, u_m, v$ be the neighbours of x in their natural cyclic order around x. As G is an inner triangulation, $uu_1u_2 \cdots u_m, v = P$ is a path. Hence the graph $G - x$ has $F' = P \cup (F - x)$ as outer face.

Assume first that $z \notin \{u,v\}$. Then let L' be the list assignment on $G - x$ defined by $L'(w) = L(w) \setminus L(x)$ if $w \in N_G(x)$ and $L'(w) = L(w)$ otherwise. Clearly, $|L'(w)| \geq 3$ if $w \in F'$ and $|L'(w)| \geq 5$ otherwise. Hence, by Theorem 2, $G - x$ admits an L'-colouring. Colouring x with the colour of its list, we obtain an L-colouring of G.

Assume now that $z \in \{u,v\}$, say $z = u$. Let α be a colour of $L(z) \setminus (L(x) \cup L(y))$. Let L' be the list assignment on $G - x$ defined by $L'(z) = \{\alpha\}$, $L'(w) = L(w) \setminus L(x)$ if $w \in N_G(x) \setminus \{z\}$ and $L'(w) = L(w)$ otherwise. Clearly, L' is (y,z,v)-correct. Hence, by the induction hypothesis, $G - x$ admits an L'-colouring. Colouring x with the colour of its list, we obtain an L-colouring of G. □

3.2 Nice, great and good paths

Let G be a graph and H an induced subgraph of G.

We denote by Z_H the set of vertices of G which are adjacent to at least 3 vertices of H. For every vertex v in $V(G)$, we denote by $N_H(v)$ the set of vertices of H adjacent to v, and we set $d_H(v) = |N_H(v)|$.

Let L be a list assignment of G. For any L-colouring ϕ of H, we denote by L_ϕ the list assignment of $G - H$ defined by $L_\phi(z) = L(z) \setminus \phi(N_H(z))$. A vertex $z \in V(G - H)$ is *safe* (with respect to ϕ), if $|L_\phi(z)| \geq 3$. An L-colouring of H is *safe* if all vertices of $z \in V(G - H)$ are safe. Observe that if L is a 5-list assignment, then for any L-colouring ϕ of H, every vertex z not in Z_H has at most two neighbours in H and therefore $|L_\phi(z)| \geq 3$. Hence ϕ is safe if and only if every vertex in Z_H is safe.

Let $P = v_1 \cdots v_p$ be an induced path in G. For $2 \leq i \leq p-1$, we denote by $[v_i]_P$, or simply $[v_i]$ if P is clear from the context, the set $\{v_{i-1}, v_i, v_{i+1}\}$. We say that a vertex z is adjacent to $[v_i]$ if it is adjacent to all vertices in the set $[v_i]$. Note that if z is adjacent to $[v_i]$ then z is not in P as P is induced.

Lemma 6. *Let $P = v_1 \cdots v_p$ be an induced path in G, x a vertex such that $N_P(x) = [v_{i+1}]$, $1 \leq i \leq p-1$, and ϕ a colouring of $P - v_i$. If $i = 1$ or $\phi(v_{i-1}) = \phi(v_{i+1})$, then one can extend ϕ to v_i such that x is safe.*

Proof. If $\{\phi(v_{i+1}), \phi(v_{i+2})\} \not\subset L(x)$, then assigning to v_i any colour distinct from $\phi(v_{i+1})$, we get a colouring of P such that x is safe. So we may assume that $\{\phi(v_{i+1}), \phi(v_{i+2})\} \subset L(x)$.

If $\phi(v_{i+2}) \in L(v_i)$, then setting $\phi(v_i) = \phi(v_{i+2})$, we have a colouring ϕ such that x is safe. If not, there is a colour α in $L(v_i) \setminus L(x)$. Necessarily, $\alpha \neq \phi(v_{i+1})$ and so one can colour v_i with α. Doing so, we obtain a colouring such that x is safe. □

Let $P = v_1 \cdots v_p$ be an induced path. It is a *nice path* in G if the following are true.

(a) for every $z \in Z_P$, $N_P(z) = [v_i]$ for some $2 \leq i \leq p-1$;

(b) for every $2 \leq i \leq p-1$, there are at most two vertices adjacent to $[v_i]$ and, if there are two such vertices, then the number of vertices adjacent to $[v_{i-1}]$ or $[v_{i+1}]$ is at most 1.

It is a *great path* in G if is is nice and satisfies the following extra property.

(c) for any $i < j$, if there are two vertices adjacent to $[v_i]$ and two vertices adjacent to $[v_j]$, then the number of vertices adjacent to $[v_{i+1}]$ or $[v_{j-1}]$ is at most 1.

A safe colouring of a path $P = v_1 \cdots v_p$ is α-*safe* if $\phi(v_1) = \alpha$.

Lemma 7. *If P is a great path and L is a 5-list assignment of G, then for any $\alpha \in L(v_1)$, there exists an α-safe L-colouring ϕ of P.*

Proof. We prove this result by induction on p, the number of vertices of P, the result holding trivially when $p \leq 2$.

Assume now that $p \geq 3$. Since P is great then every vertex of Z_P adjacent to v_1 is also adjacent to v_2 and there are at most two vertices of Z_P adjacent to $[v_2]$.

Set $\phi(v_1) = \alpha$.

1. If there is no vertex adjacent to $[v_2]$, then by induction, for any $\beta \in L(v_2) \setminus \{\alpha\}$, there is a β-safe L-colouring ϕ of $v_2 \cdots v_p$. Since $\phi(v_1) = \alpha$, ϕ is an α-safe L-colouring of P.

2. Assume now that there is a unique vertex z adjacent to $[v_2]$.

   If $\alpha \notin L(z)$, then by Case 1, there is an α-safe L-colouring ϕ of P in $G - z$. It is also an α-safe L-colouring of P in G since z is safe as $\alpha \notin L(z)$. Hence we may assume that $\alpha \in L(z)$.

   Assume there is a colour β in $L(v_2) \setminus \{\alpha\}$. By induction there is a β-safe L-colouring ϕ of $v_2 \cdots v_p$. Since $\phi(v_1) = \alpha$, we obtain an α-safe L-colouring of P because z is safe as $\beta \notin L(z)$. Hence we may assume that $L(v_2) = L(z)$. In particular, $\alpha \in L(v_2)$. Let γ be α if $\alpha \in L(v_3)$, and a colour in $L(v_3) \setminus L(v_2)$ otherwise. We set $\phi(v_3) = \gamma$. Observe that whatever colour is assigned to v_2, the vertex z will be safe.

   2.1. Assume that no vertex is adjacent to $[v_3]$. By induction hypothesis, there is a γ-safe L-colouring ϕ of $v_3 \cdots v_p$. Choosing $\phi(v_2)$ in $L(v_2) \setminus \{\alpha, \gamma\}$, we obtain an α-safe L-colouring of P.

   2.2. Assume that exactly one vertex t is adjacent to $[v_3]$. By induction hypothesis, there is a γ-safe L-colouring ϕ of $v_3 \cdots v_p$. So far all the vertices except t will be safe. So we just need to choose $\phi(v_2)$ so that t is safe.
   Observe that if $\{\gamma, \phi(v_4)\} \not\subset L(t)$, choosing any colour of $L(v_2) \setminus \{\alpha, \gamma\}$ will do the job. So we may assume that $\{\gamma, \phi(v_4)\} \subset L(t)$. If there is a colour $\beta \in L(v_2) \setminus (L(t) \cup \{\alpha\})$, then setting $L(v_2) = \beta$ will make t safe. So we may assume that $L(v_2) \setminus \{\alpha\} \subset L(t)$ and so $L(t) = L(v_2) \cup \{\gamma\} \setminus \{\alpha\}$. Thus $\phi(v_4) \in L(v_2) \setminus \{\alpha, \gamma\}$. Then setting $\phi(v_2) = \phi(v_4)$ makes t safe.

   2.3. Assume that two vertices t_1 and t_2 are adjacent to $[v_3]$. Then no vertex is adjacent to $[v_4]$. Therefore, it suffices to prove that there is an α-safe L-colouring of $v_1v_2v_3v_4$. Indeed, if we have such a colouring ϕ, then by induction, $v_4 \cdots v_p$ admits a $\phi(v_4)$-safe L-colouring ϕ'. The union of these two colourings is an α-safe L-colouring of P.
   If there exists $\beta \in L(v_4) \cap L(v_2) \setminus \{\alpha, \gamma\}$, then setting $\phi(v_2) = \phi(v_4) = \beta$, we obtain an α-safe L-colouring of $v_1v_2v_3v_4$. Otherwise, $L(v_4) \setminus \{\gamma\}$ and $L(v_2) \setminus \{\alpha\}$ are disjoint. Hence one can choose β in $L(v_2) \setminus \{\alpha\}$ and δ in $L(v_4) \setminus \{\gamma\}$ so that $|\{\beta, \gamma, \delta\} \cap L(t_i)| \leq 2$ for $i = 1, 2$. Setting $\phi(v_2) = \beta$ and $\phi(v_4) = \delta$, we obtain an α-safe L-colouring of $v_1v_2v_3v_4$.

3. Assume that two vertices z_1 and z_2 are adjacent to $[v_2]$.

   We claim that it suffices to prove that there is an α-safe L-colouring of $v_1v_2v_3$.

   Let j be the smallest index such that no vertex is adjacent to $[v_j]$. For the definition of j, consider there is no vertex adjacent to $[v_p]$ so that $j \leq p$. By the property (c) of great path, for all $3 \leq i < j$, there is exactly one vertex z_i adjacent to $[v_i]$. For $i = 3, \ldots, j-1$, one after another, one can use Lemma 6 in the path $v_{i+1}\cdots v_1$ to extend ϕ to v_{i+1}, so that z_i is safe. Then applying induction on the path $v_j \cdots v_p$, we obtain an α-safe L-colouring. This proves the claim.

   Let us now prove that an α-safe L-colouring of $v_1v_2v_3$ exists.

   If $\alpha \notin L(z_i)$, then any α-safe L-colouring of $v_1v_2v_3$ in $G - z_i$ will be an α-safe L-colouring in G. By Case 2, one can find such a colouring in $G - z_i$, so we may assume that $\alpha \in L(z_i)$.

   If there is a colour $\beta \in L(v_2) \setminus L(z_1)$, then set $\phi(v_2) = \beta$. By Lemma 6 in the path $v_3v_2v_1$, one can choose $\phi(v_3)$ in $L(v_3)$ to obtain an α-safe L-colouring of $v_1v_2v_3$. Hence we may assume that $L(z_1) = L(v_2)$. Similarly, we may assume that $L(z_2) = L(v_2)$. Therefore, any α-safe L-colouring of $v_1v_2v_3$ in $G - z_2$ will be an α-safe L-colouring in G. We can find such a colouring using Case 2.

□

We say that an induced path $P = v_1 \cdots v_p$ is *good* path if either P is great or $p \geq 4$ and there is a vertex $z \in Z_P$ adjacent to v_1 such that $\{v_1, v_4\} \subset N_P(z) \subseteq \{v_1, v_2, v_3, v_4\}$ satisfying the following conditions:

- P is a great path in $G \setminus v_1z$.
- if two vertices distinct from z are adjacent to $[v_2]$, then $N_P(z) = \{v_1, v_3, v_4\}$ and there is no vertex adjacent to $[v_3]$; and
- if two vertices distinct from z are adjacent to $[v_3]$, then $N_P(z) = \{v_1, v_2, v_4\}$ and there is no vertex adjacent to $[v_2]$.

Note that since P is induced, then z is not in P.

Lemma 8. *If $P = v_1 \cdots v_p$ is a good path and L is a 5-list assignment of G, then there exists a safe L-colouring of P.*

Proof. If P is great, then the result follows from Lemma 7. So we may assume that P is not great. Let z be the vertex of Z_P such that $\{v_1, v_4\} \subset N_P(z) \subseteq \{v_1, v_2, v_3, v_4\}$.

If there is a colour $\alpha \in L(v_1) \setminus L(z)$, then let $\phi(v_1) = \alpha$ and use Lemma 7 to colour $v_1 \cdots v_p$ in $G \setminus v_1z$. The obtained colouring ϕ is a safe L-colouring of P. For any $z' \in Z_P \setminus \{z\}$, we have $|L_\phi(z')| \geq 3$ because z' has the same neighbourhood in G and $G \setminus v_1z$. Now $|L_\phi(z)| \geq 3$ since $\alpha \notin L(z)$, so ϕ is safe. Henceforth, we assume that $L(v_1) = L(z)$.

1. Assume first that $N_P(z) = \{v_1, v_2, v_3, v_4\}$.

   By the properties of a good path, at most one vertex z' different from z is adjacent to $[v_2]$.

   1.1. Assume first that z is the unique vertex adjacent to $[v_3]$.

   If there is a colour $\alpha \in L(z) \cap L(v_3)$, then set $\phi(v_1) = \phi(v_3) = \alpha$. By Lemma 7, one can extend ϕ to $v_3 \cdots v_p$ so that all vertices of Z_P but z are safe. Then by Lemma 6 applied to

$v_2 \cdots v_p$, one can choose $\phi(v_2) \in L(v_2)$ so that z is safe for $P - v_1$. Since $\phi(v_1) = \phi(v_3)$, then ϕ is a proper colouring and z is safe for P. Hence ϕ is a safe L-colouring of P. So we may assume that $L(z) \cap L(v_3) = \emptyset$.

If there exists $\beta \in L(v_2) \setminus L(z)$, then set $\phi(v_2) = \beta$. By Lemma 7, one can extend ϕ to $v_2 \cdots v_p$ so that all vertices of Z_P but z and z' are safe. Observe that necessarily z will be safe because $\phi(v_2) \notin L(z)$ and $\phi(v_3) \notin L(z)$. By Lemma 6, one can extend ϕ to v_1 so that z' is safe, thus getting a safe L-colouring of P. So we may assume that $L(v_2) = L(z)$.

We have $|L(v_2) \cup L(v_3)| = 10 \geq |L(z')|$. So we can find $\alpha \in L(v_2)$ and $\beta \in L(v_3)$ so that $|\{\alpha, \beta\} \cap L(z')| \leq 1$. Using Lemma 7 take a β-safe L-colouring ϕ of the path $v_3 v_4 \ldots v_p$ and set $\phi(v_2) = \alpha$. If $\phi(v_4) \in L(z) \setminus \{\alpha\}$, then colour v_1 with $\phi(v_4)$, otherwise colour it with any colour distinct from α. This gives a safe L-colouring of P.

1.2 Assume now that a vertex $y \neq z$ is adjacent to $[v_3]$.

- $*$ Suppose that a vertex t is adjacent to $[v_4]$. Then z' does not exist.

  If there is a colour $\alpha \in L(v_2) \setminus L(z)$, then using Lemma 7 take an α-safe L-colouring ϕ of $v_2 \cdots v_p$. If $\phi(v_3) \notin L(z)$, then z would be safe whatever colour we assign to v_1, so there is a safe L-colouring of P. If If $\phi(v_3) \in L(z)$, then setting $\phi(v_1) = \phi(v_3)$, we obtain a safe L-colouring of P. So we may assume that $L(v_2) = L(z)$.

  If there is a colour α in $L(z) \cap L(v_4)$, then set $\phi(v_2) = \phi(v_4) = \alpha$. Then y will be safe. Extend ϕ to $v_4 \cdots v_p$ by Lemma 7. Then all the vertices are safe except t and z. By Lemma 6, one can choose $\phi(v_3)$ so that t is safe. If $\phi(v_3) \in L(z)$, then setting $\phi(v_1) = \phi(v_3)$, we get a safe L-colouring of P. If $\phi(v_3) \notin L(z)$, then whatever colour we assign to v_1, we obtain a safe colouring of P. Hence we may assume that $L(z) \cap L(v_4) = \emptyset$. By Lemma 7, there is a safe L-colouring of P in $G \setminus zv_4$. This colouring is also a safe colouring of P in G, since $\phi(v_4)$ is not in $L(z)$.

- $*$ If no vertex is adjacent to $[v_4]$, then z' may exist. In this case, it is sufficient to prove that there exists a safe L-colouring of $v_1 v_2 v_3 v_4$. Indeed, if there is such a colouring ϕ, then by Lemma 7, it can be extended to a safe L-colouring of P.

  Symmetrically to the way we proved the result when $L(v_1) \neq L(z)$, one can prove it when $L(v_4) \neq L(z)$. Hence we may assume that $L(v_4) = L(z)$.

  Assume that there is a colour $\alpha \in L(v_2) \cap L(z)$. Set $\phi(v_2) = \phi(v_4) = \alpha$. If there is a colour $\beta \in L(v_3) \setminus L(z)$, then set $\phi(v_3) = \beta$ so that z will be safe and extend ϕ with Lemma 6 so that z' is safe to obtain a safe colouring of $v_1 v_2 v_3 v_4$ in G. If $L(v_3) = L(z)$, then assign to v_1 and v_3 a same colour in $L(z) \setminus \{\alpha\}$ to get a safe colouring of $v_1 v_2 v_3 v_4$. Hence we may assume that $L(v_2) \cap L(z) = \emptyset$. Symmetrically, we may assume that $L(v_3) \cap L(z) = \emptyset$. By Lemma 7, there exists a safe colouring ϕ of $v_1 v_2 v_3 v_4$ in $G - z$. It is also a safe colouring of $v_1 v_2 v_3 v_4$ in G because $\phi(v_2)$ and $\phi(v_3)$ cannot be in $L(z)$.

2. Assume now that $N_P(z) = \{v_1, v_3, v_4\}$.

   If no vertex is adjacent to $[v_2]$, then using Lemma 7 take a safe L-colouring of $v_2 \ldots v_p$. If $\phi(v_3) \in L(z)$, then set $\phi(v_1) = \phi(v_3)$. If not colour v_3 with any colour in $L(z) \setminus \{\phi(v_2)\}$. This gives a safe L-colouring of P. Hence we may assume that a vertex t is adjacent to $[v_2]$.

   By the properties of a good path, we know that at most one vertex, say u, is adjacent to v_3. If $L(v_3) \cap L(z)$ is empty, then any safe L-colouring of P given by Lemma 7 in $G \setminus zv_1$ would be a safe L-colouring of P. Hence we may assume that there is a colour α in $L(v_3) \cap L(z)$. Set $\phi(v_1) = \phi(v_3) = \alpha$ and apply Lemma 7 to $v_3 \ldots v_p$. Then by Lemma 6, we can choose $\phi(v_2)$ so that the possible vertex u is safe. This gives a safe colouring of P.

3. Assume that $N_P(z) = \{v_1, v_2, v_4\}$.

   Suppose no vertex is adjacent to $[v_2]$. By Lemma 7, there is a safe$L$- colouring of $v_2 \dots v_p$. Set $\phi(v_1) = \phi(v_4)$ if $\phi(v_4) \in L(z) \setminus \{\phi(v_2)\}$, and let $\phi(v_1)$ be any colour of $L(v_1) \setminus \{\phi(v_2)\}$ otherwise. Doing so z is safe and so ϕ is a safe L-colouring of P. Hence we may assume that a vertex u is adjacent to $[v_2]$. By definition of good path, it is the unique vertex adjacent to $[v_2]$.

   Suppose that there exists a colour β in $L(v_2) \setminus L(z)$. By Lemma 7, there is a safe colouring ϕ of $v_2 \dots v_p$ such that $\phi(v_2) = \beta$. By Lemma 6, it can be extended to v_1 so that u is safe. This yields a safe L-colouring of P. Hence we may assume that $L(v_2) = L(z)$.

   If $L(v_4) \cap L(z) = \emptyset$, then in every colouring of P, the vertex z will be safe. Hence any safe colouring of P in $G - z$, (there is one by Lemma 7) is a safe L-colouring of P in G. So we may assume that there exists a colour $\alpha \in L(v_4) \cap L(z)$.

   Assume that at most one vertex s is adjacent to $[v_4]$. Set $\phi(v_2) = \phi(v_4) = \alpha$ so that z and all the vertices adjacent to $[v_3]$ will be safe. By Lemma 7, there is an α-safe colouring of $v_4 \dots v_p$. Now by Lemma 6, one can extend ϕ to v_3 so that s (if it exists) is safe, and then again by Lemma 6 extend it to v_1 so that u is safe. This gives a safe L-colouring of P. So we may assume that two vertices s and s' are adjacent to $[v_4]$.

   Assume that there is a vertex t adjacent to $[v_3]$, then there is no vertex adjacent to $[v_5]$. Hence it suffices to find a safe L-colouring of $v_1v_2v_3v_4v_5$. Indeed, if we have such a colouring ϕ, then using Lemma 7, one can extend it to a safe L-colouring of P. Set $\phi(v_2) = \phi(v_4) = \alpha$. Doing so t and z will be safe. If α or some colour $\beta \in L(v_5) \setminus \{\alpha\}$ is not contained in one of lists $L(s)$ and $L(s')$, say $L(s')$. Then colouring v_5 with β, if it exists, or any other colour otherwise, the vertex s' will also be safe. By Lemma 6, one can colour v_3 so that s is safe. By Lemma 6, one can then colour v_1 to obtain a colouring for which u is safe. This L-colouring of $v_1v_2v_3v_4v_5$ is safe. Hence, we may assume that $L(s) = L(s') = L(v_5)$. Colour v_5 with any colour in $L(v_5) \setminus \{\alpha\}$. Using Lemma 6, colour v_3 so that s is safe. Then s' will be also safe because $L(s) = L(s')$. Again by Lemma 6, colour v_1 so that u is safe to obtain a safe colouring of $v_1v_2v_3v_4v_5$.

   Assume finally that no vertex is adjacent to $[v_3]$. By Lemma 7, there is a safe L-colouring ϕ of $v_3 \dots v_p$. If $\phi(v_4) \notin L(z)$, then assign to v_2 any colour in $L(v_2) \setminus \{\phi(v_3)\}$. If not, then set $\phi(v_2) = \phi(v_4)$. (This is possible since $L(v_2) = L(z)$.) Then z will be safe. By Lemma 6, colour v_1 so that u is safe to obtain a safe L-colouring of P.

□

3.3 Main theorem

A drawing of G is *nice* if two edges intersect at most once. It is well known that every graph with crossing number k has a nice drawing with at most k crossings. (See [5] for example.) In this paper, we will only consider nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will confound a crossing with this set of two edges. The *cluster* of a crossing C is the set of endvertices of its two edges and is denoted $V(C)$.

Theorem 9. *Let G be a graph having a drawing in the plane with two crossings. Then* $\mathrm{ch}(G) \leq 5$.

Proof. By considering a counter-example G with the minimum number of vertices. Let L be a 5-list assignment of G such that G is not L-colourable.

Let C_1 and C_2 be the two crossings. By Theorem 3, C_1 and C_2 have no edge in common. Set $C_i = \{v_iw_i, t_iu_i\}$. Free to add edges and to redraw them along the crossing, we may assume that v_iu_i, u_iw_i, w_it_i and t_iv_i are edges and that the 4-cycle $v_iu_iw_it_i$ has no vertex inside but the two edges of C_i. In addition, we assume that $u_1v_1t_1w_1$ appear in clockwise order around the crossing point of C_1 and that $u_2v_2t_2w_2$ appear in counter-clockwise order around the crossing point of C_2. Free to add edges, we may also assume that $G \setminus \{v_1w_1, v_2w_2\}$ is a triangulation of the plane. In the rest of the proof, for convenience, we will refer to this fact by writing that G is *triangulated*.

Claim 9.1. *Every vertex of G has degree at least* 5.

Proof. Suppose not. Then G has a vertex x of degree at most 4. By minimality of G, $G - x$ has an L-colouring ϕ. Now assigning to x a colour in $L(x) \setminus \phi(N(x))$ we obtain an L-colouring of G, a contradiction. □

A cycle is *separating* if none of its edges is crossed and both its interior and exterior contain at least one vertex. A cycle is *nicely separating* if it is separating and its interior or its exterior has no crossing.

Claim 9.2. *G has no nicely separating triangle.*

Proof. Assume, by way of contradiction, that a triangle $T = x_1x_2x_3$ is nicely separating. Let G_1 (resp. G_2) be the subgraph of G induced by the vertices on T or outside T (resp. inside T). Without loss of generality, we may assume that G_2 is a plane graph.

By minimality of G, G_1 has an L-colouring ϕ_1. Let L_2 be the list assignment of G_2 defined by $L_2(x_1) = \{\phi_1(x_1)\}$, $L_2(x_2) = \{\phi_1(x_1), \phi_1(x_2)\}$, $L_2(x_3) = \{\phi_1(x_1), \phi_1(x_2), \phi_1(x_3)\}$, and $L_2(x) = L(x)$ for every vertex inside T. Then L_2 is a suitable list assignment of G_2, so by Theorem 2, G_2 admits an L_2-colouring ϕ_2. Observe that necessarily $\phi_2(x_i) = \phi_1(x_i)$. Hence the union of ϕ_1 and ϕ_2 is an L-colouring of G, a contradiction. □

Claim 9.3. *Let $C = abcd$ be a* 4*-cycle with no crossing inside it. If a and c have no common neighbour inside C then C has no vertex in its interior.*

Proof. Assume by way of contradiction that the set S of vertices inside C is not empty.

Then ac is not an edge otherwise one of the triangles abc and acd would be nicely separating. Since G is triangulated, the neighbours of a (resp. c) inside C plus b and d (in cyclic order around a (resp. c)) form a (b,d)-path P_a (resp. P_c). The paths P_a and P_c are internally disjoint because a and c have no common neighbour inside C. Hence $P_a \cup P_c$ is a cycle C'. Furthermore C' is the outerface of $G' = G\langle S \cup \{b,d\}\rangle$.

By minimality of G, $G_1 = (G - S) \cup bd$ admits an L-colouring ϕ. Let L' be the list-colouring of G' defined by $L'(b) = \{\phi(b)\}$, $L'(d) = \{\phi(d)\}$, $L'(x) = L(x) \setminus \{\phi(a)\}$ if x is an internal vertex of P_a, $L'(x) = L(x) \setminus \{\phi(c)\}$ if x is an internal vertex of P_c, and $L'(x) = L(x)$ if $x \in V(G' - C')$. Then L' is a $\{b,d\}$-correct list assignment of G'. Hence, by Lemma 5, G' admits an L'-colouring ϕ'. The union of ϕ and ϕ' is an L-colouring of G, a contradiction. □

Claim 9.4. *G has no nicely separating* 4*-cycle.*

Proof. Suppose not. Then there exists a nicely separating 4-cycle $abcd$. Let $b = z_1, z_2, \dots, z_{p+1} = d$ be the common neighbours of a and c in clockwise order around a. By Claim 9.3, we have $p \geq 2$. Each of the 4-cycles az_icz_{i+1}, $1 \leq i \leq p$ has empty interior by Claim 9.3. So z_2 has degree at most 4. This contradicts Claim 9.1. □

A path P is *friendly* if there are two adjacent vertices x and y such that $|N_P(x)| \leq 4$, $|N_P(y)| \leq 3$ and P is good in $G - \{x,y\}$. A path P *meets* a crossing if it contains at least one endvertex of each of the two crossed edges. A *magic path* is a friendly path meeting both crossings.

Claim 9.5. *G has no magic path Q.*

Proof. Suppose for a contradiction that G has a magic path Q. Then there exists two adjacent vertices x and y such that $|N_Q(x)| \leq 4$, $|N_Q(y)| \leq 3$ and P is good in $G - \{x,y\}$. Lemma 8, there in a L-colouring ϕ of Q such that every vertex z of $(G-Q) - \{x,y\}$ satisfies $|L_\phi(z)| \geq 3$. Now $|L_\phi(x)| \geq 1$ and $|L_\phi(y)| \geq 2$, because $|N_Q(x)| \leq 4$ and $N_Q(y) \leq 3$ Since Q meets the two crossings, $G-Q$ is planar. Furthermore, $G-Q$ may be drawn in the plane such that all the vertices on the outer face are those of $N(Q)$. Hence L_ϕ is a suitable assignment of $G-Q$. Hence by Theorem 2, $G-Q$ is L_ϕ-colourable and so G is L-colourable, a contradiction. □

In the remaining of the proof, we shall prove that G contains a magic path, thus getting a contradiction. Therefore, we consider *shortest* (C_1,C_2)*-paths*, that are paths joining C_1 and C_2 with the smallest number of edges. We first consider the cases when the distance between C_1 and C_2 is 0 or 1. We then deal with the general case when $dist(C_1,C_2) \geq 2$.

Claim 9.6. *$dist(C_1,C_2) > 0$.*

Proof. Assume for a contradiction that $dist(C_1,C_2) = 0$. Then, without loss of generality, $v_1 = v_2$. Note that $u_1 \neq u_2$ as otherwise the path u_1v_1 would be magic, contradicting Claim 9.5. Similarly, we have $t_1 \neq t_2$.

Note that w_1 is not adjacent to u_2 for otherwise both the interior and exterior of $w_1u_1v_1u_2$ would contain at least one neighbour of u_1 by Claim 9.1. Thus this 4-cycle would be nicely separating, a contradiction to Claim 9.4. Henceforth, by symmetry, w_1 is not adjacent to u_2 nor t_2 and w_2 is not adjacent to u_1 nor t_1.

If u_1 is not adjacent to u_2, then consider the induced path $Q = u_1v_1u_2$. Since w_1 and w_2 are not adjacent to u_2 and u_1, respectively, then $\{w_1,w_2\} \cap Z_Q = \emptyset$. The vertices t_1 and $t_2$cannot be both in Z_Q for otherwise u_1t_2 and u_2t_1 would cross. Furthermore, if z_1 and z_2 are distinct vertices in $Z_Q \setminus \{t_1,t_2\}$, then either $u_1v_1u_2z_1$ nicely separates z_2 or $u_1v_1u_2z_2$ nicely separates z_1 contradicting Claim 9.4. Thus, $|Z_Q| \leq 2$ and Q is magic contradicting Claim 9.5. Henceforth, u_1 is adjacent to u_2, and, by a symmetrical argument, t_1 is adjacent to t_2.

If u_1 is adjacent to t_2, then both the interior and exterior of $u_1u_2w_2t_2$ contain at least one neighbour of w_2 by Claim 9.1. Thus this 4-cycle would be nicely separating, a contradiction to Claim 9.4. Henceforth, u_1 is not adjacent to t_2, and symmetrically t_1 is not adjacent to u_2.

Therefore $Q = u_1v_1t_2$ is an induced path. Note that $Z_Q \subseteq N(v_1)$. The triangles $v_1u_1u_2$ and $v_1t_1t_2$ together with Claim 9.2 imply that $N(v_1) = \{u_1,u_2,t_1,t_2,w_1,w_2\}$. Since w_1 is not adjacent to t_2 and w_2 is not adjacent to u_1, then $Z_Q = \{u_2,t_1\}$. Thus Q is magic contradicting Claim 9.5. □

Claim 9.7. *Let $i \in \{1,2\}$ and x a vertex not in C_i. Then at most one vertex in $\{u_i,t_i\}$ is adjacent to x and at most one vertex in $\{v_i,w_i\}$ is adjacent to x.*

Proof. Assume for a contradiction that x is adjacent to both u_i and t_i. Observe that the edges u_ix and t_ix are not crossed since $dist(C_1,C_2) \geq 1$. Then one of the two 4-cycles $u_iv_it_ix$ and $u_iw_it_ix$ is nicely separating. Thus the region bounded by this cycle has no vertex by Claim 9.4. Hence either $d(v_i) \leq 4$ or $d(w_i) \leq 4$. This contradicts Claim 9.1.

Similarly, one shows that at most one vertex in $\{v_i,w_i\}$ is adjacent to x. □

Claim 9.8. $dist(C_1,C_2) > 1$.

Proof. Assume for a contradiction that $dist(C_1,C_2) = 1$. Without loss of generality, we may assume that $v_1v_2 \in E(G)$.

Let us first show that without loss of generality, we may assume that u_1 is not adjacent to v_2 and u_2 is not adjacent to v_1. By symmetry, if t_1 is not adjacent to v_2 and t_2 is not adjacent to v_1, then we get the result by renaming swapping the names of u_i and t_i, $i = 1,2$. Thus by symmetry and by Claim 9.7, if it not the case, then $u_1v_2 \in E(G)$ and $v_1t_2 \in E(G)$. Moreover w_1v_2 is not an edge by Claim 9.7. Hence renaming u_1, v_1, t_1, w_1 into v_1, t_1, w_1, u_1 respectively, we are in the desired configuration.

The vertices u_1 and u_2 are not adjacent, for otherwise the cycle $u_1v_1v_2u_2$ would be nicely separating since G is triangulated and u_1v_2 and u_2v_1 are not edges. So Q is an induced path.

A vertex of Z_Q is *goofy* if it is adjacent to u_1 and u_2.

- Suppose first that there is a goofy vertex z' not in $C_1 \cup C_2$.

  Without loss of generality, we may assume that z' is adjacent to u_1, v_1 and u_2. If the crossing C_1 is inside $z'u_1v_1$, then consider the path $R = t_1v_1v_2u_2$. It is induced since $z'u_1v_1$ separates t_1 from v_2 and u_2. Moreover all the neighbours of t_1 are inside $z'u_1v_1$, so they have at most two neighbours in R except for u_1 which is not adjacent to v_2 nor to u_2. Hence the vertices of Z_R are all adjacent to $\{v_1,v_2,u_2\}$. Moreover $w_2 \notin Z_R$ because w_2v_1 is not an edge by Claim 9.7. Hence by planarity of $G - \{w_1,w_2\}$, there are at most two vertices adjacent to $\{v_1,v_2,u_2\}$. Thus R is magic, a contradiction.

  Hence we may assume that C_1 is outside $z'u_1v_1$. The 4-cycle $z'v_1v_2u_2$ is not nicely separating by Claim 9.4, and G is triangulated. So $z'v_2 \in E(G)$ because v_1 is not adjacent to u_2. So z' is adjacent to all vertices of Q.

  Then there is no other vertex z'' in $Z_Q \setminus \{C_1 \cup C_2\}$, for otherwise one of the crossing C_i is inside u_iv_iz'' and as above, we obtain the contradiction that R is magic.

  Now w_1u_2 is not an edge, for otherwise $w_1u_1z'u_2$ would be separating since $d(u_1) \geq 5$, a contradiction to Claim 9.4. Similarly, w_2u_1 is not an edge. Hence $Z_Q \subset \{z',t_1,t_2\}$. Now one of the edges t_1u_2 and t_2u_1 is not in $E(G)$, since otherwise they would cross. Without loss of generality, t_1 is not adjacent to u_2. Then Q is good in $G - t_2$, and so Q is magic. This contradicts Claim 9.5.

- Suppose now that all the goofy vertices of Z_Q are in $C_1 \cup C_2$.

  Suppose first that w_1 is in Z_Q, then w_1u_2 is an edge because w_1 is not adjacent to v_2 according to Claim 9.7. Thus t_2 and w_2 are not adjacent to u_1. So $w_2 \notin Z_Q$ and $N_Q(t_2) \subset \{v_1,v_2,u_2\}$, so t_2 is not goofy. Moreover by planarity of $G - \{w_1,w_2\}$, there is at most two vertices adjacent $\{v_1,v_2,u_2\}$. Furthermore, all the vertices distinct from t_1 and adjacent to $\{u_1,v_1,v_2\}$ are in the region bounded by $w_1v_1v_2u_2$ containing u_1. Therefore there is at most one such vertex. Hence Q is good in $G - \{w_1,t_1\}$. Thus Q is magic and contradicts Claim 9.5.

  Similarly, we get a contradiction if $w_2 \in Z_Q$. So $Z_Q \cap (C_1 \cup C_2) \subseteq \{t_1,t_2\}$. Then easily Q is good in $G - t_2$ and so Q is magic. This contradicts Claim 9.5.

□

Claim 9.9. *Some of the shortest (C_1,C_2)-paths is nice.*

Proof. Let $P = x_1x_2\cdots x_p$ be any shortest (C_1,C_2)-path. Then no vertex in C_1 is adjacent to a vertex in $P-\{x_1,x_2\}$. Therefore, $V(C_1)\cap Z_P = \emptyset$. Similarly, we have $V(C_2)\cap Z_P = \emptyset$. Hence the graph G' induced by $V(P)\cup Z_P$ is planar as it contains exactly one vertex from each crossing.

Any vertex not in P can be adjacent only to vertices of P at distance at most two from each other, otherwise there would be a (C_1,C_2)-path shorter than P. Thus, if $z \in Z_P$, then z has precisely three neighbours in P. Moreover, there exists an $i \in \{2,\dots,p-1\}$ such that $N_P(z) = [x_i]$.

If there are distinct vertices $z_1,z_2,z_3 \in Z_P$ such that $N_P(z_1) = N_P(z_2) = N_P(z_2) = [x_i]$ for some value of i, then the subgraph of G' induced by $\{z_1,z_2,z_3\}\cup\{x_{i-1},x_i,x_{i+1}\}$ contains a $K_{3,3}$. By Kuratowski's Theorem, this contradicts the fact that G' is planar. Therefore, for every $2 \le i \le p-1$, there are at most two vertices in Z_P adjacent to $[x_i]$.

Let $z_1,z_2 \in Z_P$ be such that $N_P(z_1) = N_P(z_2) = [x_i]$. The edges of $H = G[\{z_1,z_2\}\cup[x_i]]$ separate the plane into five regions $R_1,\dots,R_5$ as follows. Let R_1 be the region bounded by $x_{i-1}x_iz_1$ not containing the vertex z_2, R_2 be the region bounded by $x_ix_{i+1}z_1$ not containing the vertex z_2, R_3 be the region bounded by $x_{i-1}x_iz_2$ not containing the vertex z_1, R_4 be the region bounded by $x_ix_{i+1}z_2$ not containing the vertex z_1 and R_5 be the region bounded by $x_{i-1}z_1x_{i+1}z_2$ not containing x_i (see Figure 1). Since $(V(C_1)\cup V(C_2))\cap Z_P = \emptyset$ and P is a shortest (C_1,C_2)-path, then no edge in H is crossed.

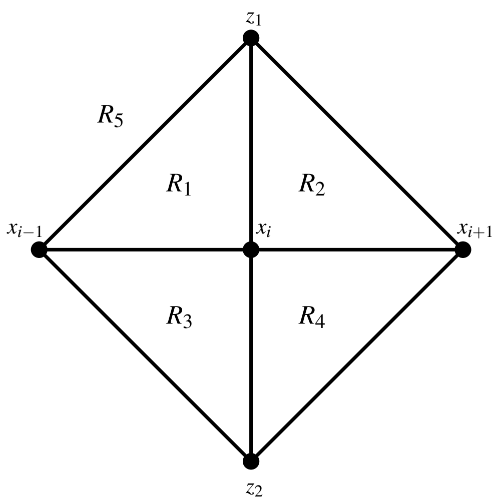

Figure 1: Regions R_1, R_2, R_3, R_4 and R_5.

Let J_P be the subset of $\{3,\dots,p-2\}$ such that for $j \in J_P$, there are two vertices in Z_P adjacent to $[x_j]$ and at least one vertex adjacent to $[x_{j-1}]$ and another adjacent to $[x_{j+1}]$. The path P is said to be *semi-nice* if $J_P = \emptyset$.

Let us first prove that some of the shortest (C_1,C_2)-paths is semi-nice.

> Suppose for a contradiction that no shortest (C_1,C_2)-path is semi-nice. Let P be a shortest (C_1,C_2)-path that maximizes the smallest index i in J_P. Let $z_1,z_2 \in Z_P$ be such that $N_P(z_1) = N_P(z_2) = [x_i]$.
>
> Let $z \in Z_P$ be a vertex adjacent to $[x_{i+1}]$. If C_2 is in R_5, then so is x_{i+2} and we get a contradiction from the fact that either zx_i or zx_{i+2} must cross an edge of H. Since P defines a path between x_{i+1} and $V(C_2)$, then C_2 must be either in R_2 or in R_4 (say R_4). Similarly, C_1 is either in R_1 or in R_3. The cycle $x_{i-1}x_ix_{i+1}z_2$ is not be a nicely separating cycle by Claim 9.4, so C_1 must be in R_1.

Now, by Claim 9.2, R_2 and R_3 are empty, and, by Claim 9.4, there is no vertex in R_5. Since P is a shortest path, $x_{i-1}x_{i+1}$ is not an edge and therefore z_1 is adjacent to z_2 as G is triangulated.

Now, consider the path P' obtained from P by replacing x_i with $x'_i = z_2$. Note that P' is also a shortest path and that both z_1 and x_i are adjacent to $[x'_i]$. Since no edge in H is crossed, for any $v \in V(G) \setminus (\{z_1,z_2\} \cup [x_i])$, if v is adjacent to x_{i-1} then it must be in R_1 and if v is adjacent to z_2 then it must be in R_4. Therefore, there is no vertex in $Z_{P'}$ adjacent to $\{x_{i-2},x_{i-1},z_2\}$. This implies that if $j \in J_{P'}$, then either $j \leq i-3$ or $j \geq i+1$. Note that if $j \in J_{P'}$ and $j \leq i-3$, then $j \in J_P$. As i is the minimum of J_P, the minimum of $J_{P'}$ is at least $i+1$. This contradicts our choice of P.

Let K_P be the subset of $\{2,\dots,p-1\}$ such that for $k \in K_P$, there are two vertices in Z_P adjacent to $[x_k]$ and two vertices adjacent to $[x_{k+1}]$. Observe that a nice path P is a semi-nice path such that K_P is empty, that is a path such that J_P and K_P are empty.

Suppose, by way of contradiction, that every (C_1,C_2)-shortest path is not nice. Then consider the semi-nice (C_1,C_2)-shortest path that maximizes the minimum of K_P.

Let $z_1,z_2,z_3,z_4 \in Z_P$ be such that $N_P(z_1) = N_P(z_2) = [x_i]$ and $N_P(z_3) = N_P(z_4) = [x_{i+1}]$, where i is the smallest index in K_P. Recall that the edges of $H = G[\{z_1,z_2\} \cup [x_i]]$ separate the plane into the five above-described regions $R_1,\dots,R_5$. Again, we can use z_3 or z_4 to prove that C_2 is either in R_2 or in R_4 (say R_4). Therefore, x_{i+2} is in R_4 which implies z_3 and z_4 are also in R_4. Thus, z_1 is not adjacent to z_3 nor z_4. Furthermore, z_2 cannot be adjacent to both z_3 and z_4 for otherwise we can obtain a K_5 in the subgraph of G' induced by $[x_{i+1}] \cup \{z_2,z_3,z_4\}$ by contracting the edge z_4x_{i+2} (see Figure 2). Thus, without loss of generality, suppose z_2 and z_3 are not adjacent.

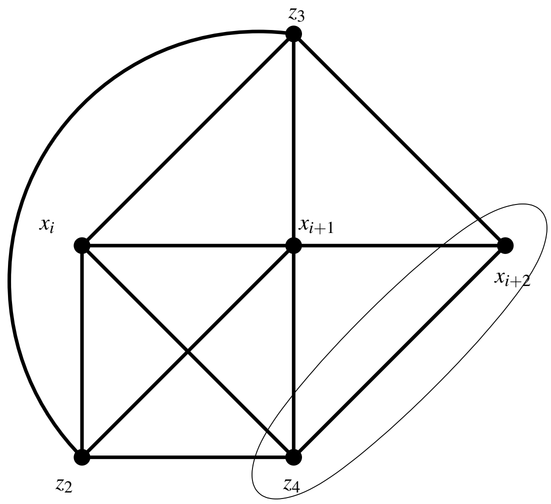

Figure 2: K_5 minor of G' is obtained by contracting z_4x_{i+2}.

Consider the path P' obtained from P by replacing x_{i+1} with $x'_{i+1} = z_3$. Since no edge in H is crossed, for any $v \in V(G) \setminus (\{z_1,z_2\} \cup [x_i])$, if v is adjacent to x_{i-1} then it is not in R_4, and if v is adjacent to z_3 then it must be in R_4. Since neither z_1 nor z_2 are adjacent to z_3 and x_{i+1} is not adjacent to x_{i-1}, there is no vertex in $Z_{P'}$ adjacent to $\{x_{i-1},x_i,z_3\}$. This implies that if $k \in K_{P'}$, then either $k \leq i-2$ or $k \geq i+1$. Note that if $k \in K_{P'}$ and $k \leq i-2$, then $k \in K_P$. This implies that the minimum index in $K_{P'}$ is strictly greater than i. Hence by our choice of P, the path P' is not semi-nice, that is $J_{P'} \neq \emptyset$.

Observe that if $j \in J_{P'}$, then either $j \leq i-2$ or $j \geq i+2$. Note that if $j \in J_{P'}$ and either $j \leq i-2$ or $j \geq i+4$, then $j \in J_P$. Since J_P is empty, then $J_{P'} \subseteq \{i+2, i+3\}$. Let $z'_1, z'_2 \in Z_{P'}$ be such that $N_{P'}(z'_1) = N_{P'}(z'_2) = [x'_j]$, for some $j \in J_{P'}$ with $J_{P'} \subseteq \{i+2, i+3\}$. Note that for the two possible values of j, both z'_1 and z'_2 are adjacent to x_{i+3}. Since P is a shortest (C_1,C_2)-path, neither z_2 nor x_{i+1} are adjacent to x_{i+3} and therefore z'_1 and z'_2 are in R_4. Let R'_1 be the region bounded by $x'_{j-1}x'_jz'_1$ not containing the vertex z'_2 and R'_3 be the region bounded by $x'_{j-1}x'_jz'_2$ not containing the vertex z'_1. Both of these regions are contained in R_4. With the same argument used above in the proof of existence of a semi-nice path, one shows that if $j \in J_{P'}$, then C_1 is either contained in R'_1 or in R'_3. We get a contradiction as the path P from $V(C_1)$ to x_{i-1} crosses an edge of H. □

Claim 9.10. *There exists an induced path $Q = x_0x_1 \cdots x_px_{p+1}$ with the following properties:*

- P_1. *$P = x_1 \cdots x_p$ is a shortest (C_1,C_2)-path and is a nice path;*
- P_2. *$x_0 \in V(C_1)$ and $x_{p+1} \in V(C_2)$ but x_0x_1 and x_px_{p+1} are not crossed edges; and*
- P_3. *there is at most one vertex in Z_Q adjacent to both vertices in $\{x_0,x_3\}$ and at most one vertex in Z_Q adjacent to both vertices in $\{x_{p-2},x_{p+1}\}$.*
- P_4. *for any $i < j$, if there are two vertices adjacent to $[v_i]$ and two vertices adjacent to $[v_j]$, then the number of vertices adjacent to $[v_{i+1}]$ or to $[v_{j-1}]$ is at most 1.*

Proof. By Claim 9.9 there exists a shortest (C_1,C_2)-path $P = x_1 \cdots x_p$ which is nice. Without loss of generality, we may assume that $x_1 = v_1$ and $x_p = v_2$. According to Claim 9.7, we can choose vertices $x_0 \in \{u_1,t_1\}$ and $x_{p+1} \in \{u_2,t_2\}$ such that Q is induced. Therefore, we have at least one path satisfying properties P_1 and P_2. We say that x_0 is a *valid endpoint* if there is at most one vertex in Z_Q adjacent to both vertices in $\{x_0,x_3\}$ and x_{p+1} is a *valid endpoint* if there is at most one vertex in Z_Q adjacent to both vertices in $\{x_{p-2},x_{p+1}\}$.

Let Q be a path satisfying properties P_1 and P_2 which maximizes the number of valid endpoints of Q.

Let us first show that Q has only valid endpoints, and satisfies property P_4. By contradiction, suppose that Q has an invalid endpoint. Without loss of generality, x_0 is invalid.

Let $z_1,z_2 \in Z_Q$ be two vertices adjacent to both vertices in $\{x_0,x_3\}$. Since P is a shortest (C_1,C_2)-path, no vertex of C_1 is adjacent to x_3. Therefore, no edge of $x_0x_1x_2x_3z_1$ and $x_0x_1x_2x_3z_2$ is crossed. Let R_1 be the region bounded by $x_0x_1x_2x_3z_1$ that does not contain z_2 and R_2 be the region bounded by $x_0x_1x_2x_3z_2$ that does not contain z_1. Since the edges bounding the regions R_1 and R_2 are not crossed, then the crossing C_1 is contained in one of the regions R_1 or R_2 (say R_1). Let $\hat{x}_0$ be the vertex of $\{u_1,t_1\} \setminus \{x_0\}$ (see Figure 3).

Assume first that $\hat{x}_0$ is not adjacent to x_2. Let $\hat{Q}$ be the path obtained from Q by replacing x_0 with $\hat{x}_0$. Clearly the path $\hat{Q}$ is induced and satisfies properties P_1 and P_2. By definition of Q, $\hat{x}_0$ must be an invalid endpoint. Hence, there is a vertex $\hat{z}$ in $Z_{\hat{Q}} \setminus \{z_1\}$ which is adjacent to $\hat{x}_0$ and x_3. This vertex in necessarily inside R_1 because it is adjacent to x_0. But then, by planarity, z_1 cannot be adjacent to x_1 and x_2, a contradiction to $z_1 \in Z_Q$.

Assume now that $\hat{x}_0$ is adjacent to x_2. Let Q' be the path obtained from Q by replacing x_0 with w_1 and x_1 with $\hat{x}_0$. Note that Q' is induced as w_1 is not adjacent to x_2 by Claim 9.7.

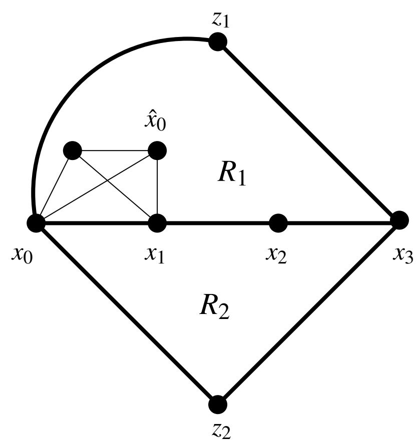

Figure 3: Regions R_1 and R_2 and the vertex $\hat{x}_0$.

Note that property P_2 is valid for Q'. The path $P' = \hat{x}_0 x_2 \cdots x_p$ is a (C_1, C_2) shortest path. Let us prove that P' is nice and so that P' satisfies property P_1. If $p = 3$, then, since no vertex in the cluster of C_1 is adjacent to x_3, at most two vertices are in $Z_{P'}$ for otherwise we would get a $K_{3,3}$ in $G - \{w_1, w_2\}$, which is impossible as this graph is planar. Thus P' is nice. Suppose now that $p \geq 4$. By planarity, z_1 is not adjacent to x_1, so z_1 is adjacent to x_2 as $z_1 \in Z_Q$. In addition, $z_1 x_2$ is contained in R_1. Thus, any vertex in $Z_{P'}$ adjacent to $\hat{x}_0$ must be in region R_1 and cannot be adjacent to x_3. Hence no vertex is adjacent to $[x_2]_{P'}$ so, since P is a nice path, P' is also a nice path.

By definition of Q, w_1 must be an invalid endpoint of Q'. Hence, there is a vertex z' in $Z_{Q'} \setminus \{z_1\}$ which is adjacent to w_1 and x_3. This vertex in necessarily inside R_1 because neither x_0 nor x_1 are adjacent to x_3. But then, by planarity, z_1 cannot be adjacent to x_1 and x_2, a contradiction to $z_1 \in Z_Q$.

Let us now prove that Q satisfies property P_4. By contradiction, suppose Q does not. Let $z_1, z_2, z'_1, z'_2 \in Z_Q$ be such that both z_1 and z_2 are adjacent to $[x_i]$ and z'_1 and z'_2 are adjacent to $[x_j]$. Consider the regions $R_1, \ldots, R_5$ related to z_1 and z_2 used in Figure 1. Consider the regions $R'_1, \ldots, R'_5$ related to z'_1 and z'_2 used in Figure 1 for $i = j$.

Let $z \in Z_Q$ be adjacent to $[x_{i+1}]$. Note that we can have $\{z_1, z_2\} \cap \{u_1, t_1\} \neq \emptyset$ if $i = 1$. But since $dist(C_1, C_2) \geq 2$, the edges $z_1 x_{i+1}$ and $z_2 x_{i+1}$ are not crossed. Furthermore, since no vertex in the cluster of C_1 is adjacent to x_3 and not vertex in the cluster of C_2 is adjacent to x_1 (P is a shortest (C_1, C_2)-path), then z is not in the cluster of either crossing.

Therefore, since z is adjacent to both x_i and x_{i+2}, we must have that both z and x_3 are in R_2 or in R_4 (say R_2). This also implies that C_2 is in R_2. Note also that, by our choice of x_0, the edges $z_1 x_i$ and $z_2 x_i$ are not crossed. Therefore, C_1 is contained in $R_1 \cup R_3 \cup R_5$. With a symmetric argument, we have that C_1 is either in R'_1 or in R'_3 (say R_1). Since both z'_1 and z'_2 are also in R_2, then $R'_1 \cup R'_3$ are contained in R_2 and we get a contradiction. □

Let Q be a path given by Claim 9.10. Without loss of generality, suppose $x_1 = v_1$ and $x_p = v_2$. Note also that Claim 9.7 implies w_1 and w_2 are not in Z_Q and therefore $G[V(Q) \cup Z_Q]$ is planar.

Claim 9.11. *$dist(C_1, C_2) = 2$ and there is a vertex adjacent to x_0 and x_4.*

Proof. Suppose not. Then no vertex in Z_Q is adjacent to vertices at distance at least four in Q. Observe that this is the case when $dist(C_1, C_2) \geq 3$, since $x_1 \ldots x_p$ is a shortest (C_1, C_2)-path.

Since P is a nice and shortest (C_1,C_2)-path, then the only vertices in Z_Q adjacent to vertices at distance at least three in Q must be adjacent to both x_0 and x_3 or to both x_{p-2} and x_{p+1}. By the property P_3 of Claim 9.10, there is at most one vertex, say z, adjacent to x_0 and x_3 and at most one vertex, say z', adjacent to x_{p-2} and x_{p+1}.

Let us make few observations.

Obs. 1 If two vertices z_1 and z_2 distinct from z are adjacent to $[x_2]$, then no vertex is adjacent to $[x_1]$ and $N_Q(z) = \{x_0,x_1,x_3\}$. Indeed z must be in the region R_5 in Figure 1 because it is adjacent to x_0 and x_3. By the planarity of $G[V(Q) \cup Z_Q]$ and since z is adjacent to x_0, x_0 must also be in R_5. Again by planarity, z is not adjacent to x_2 and, therefore, must be adjacent to x_1 as $z \in Z_Q$.

Obs. 2 If two vertices z_1 and z_2 distinct from z are adjacent to $[x_1]$, then no vertex is adjacent to $[x_2]$ and $N_Q(z) = \{x_0,x_2,x_3\}$. This argument is symmetric to Observation 1.

Suppose that z exists.
If z' exists, by Observations 1 and 2 (and their analog for z') and the properties of Q from Claim 9.10, the path Q is good in $G-z'$ because it is great in $G-\{z,z'\}$. Hence Q is magic, a contradiction to Claim 9.5. Hence z' does not exists.
By Claim 9.7, w_2 is not adjacent to x_{p-1} and w_1 is not adjacent to x_p since $dist(C_1,C_2) \geq 2$. So, by planarity of $G-\{w_1,w_2\}$, at most two vertices are adjacent to $[x_p]$. Let y be a vertex adjacent to $[x_p]$. The path Q is not great in $G-\{y,z\}$, for otherwise it would be magic. Hence, according to the properties of Q and the above observations, there must be two vertices adjacent to $[x_p]$, two vertices adjacent to $[x_{p-1}]$ and one vertex adjacent to $[x_{p-2}]$. Let z_1 and z_2 be the two vertices adjacent to $[x_{p-1}]$ and $R_1 \dots R_5$ be the regions as in Figure 1 with $i = p-1$. Since there is a vertex adjacent to $[x_{p-2}]$, then C_1 is in R_1 or R_3, and C_2 is in R_2 or R_4 because a vertex is adjacent to $[x_p]$. But by Claim 9.4 the 4-cycle $z_1x_pz_2x_{p-2}$ is not nicely separating, so there is no vertex inside R_5. Since G is triangulated, and $x_{p-2}x_p$ is not an edge because P is a shortest (C_1,C_2)-path, $z_1z_2 \in E(G)$. Now the path Q is good in $G-\{z_1,z_2\}$ and so is magic. This contradicts Claim 9.5.

Hence we may assume that z does not exists and by symmetry that z' does not exist. We get a contradiction similarly by considering a vertex w adjacent to $[x_1]$ in place of z. □

Claim 9.12. *There is precisely one vertex $z \in Z_Q$ adjacent to both x_0 and x_4.*

Proof. Observe that there are at most two vertices adjacent to x_0 and x_4. Indeed such vertices cannot be in the crossings because $dist(C_1,C_2) = 2$. Thus if there were three such vertices, together with contracting the path $x_1x_2x_3$ we would get $K_{3,3}$ minor in $G-\{w_1,w_2\}$, a contradiction.

Suppose by contradiction that two distinct vertices $z_1,z_2 \in Z_Q$ adjacent to vertices x_0 and x_4. The edges of Q are contained in the same region of the plane bounded by the cycle $x_0z_1x_4z_2$. Therefore, both crossings are also in the region containing the edges of Q. By Claim 9.3, the region bounded by the cycle $x_0z_1x_4z_2$ that does not contain the crossings has no vertex in its interior. Since G is triangulated, $z_1z_2 \in E(G)$ as x_0 because x_4 are not adjacent as $dist(C_1,C_2) = 2$.

By the property P_3 of Claim 9.10, z_1 and z_2 cannot be both adjacent to the five vertices in Q. Therefore, without loss of generality, suppose $|N_Q(z_2)| \leq 4$. Let us prove that Q is great in $H = (G-z_2) \setminus \{z_1x_0, z_1x_4\}$.

(i) If a vertex t in $G-\{z_1,z_2\}$ is adjacent to at least four vertices of Q, then without loss of generality it is adjacent to $\{x_0,x_1,x_2,x_3\}$ as it cannot be adjacent to x_0 and x_4. Now by property P_3, z_1 and z_2 are not adjacent to x_3. Hence one of them (the one such that $x_0x_1x_2x_3x_4z_i$ separates t from z_{3-i}) cannot be adjacent to any vertex of $\{x_1,x_2,x_3\}$, a contradiction to the fact that it is in Z_Q. Hence Q satisfies (a) in H.

(ii) If two vertices t_1 and t_2 of H are adjacent to $[x_2]$, then necessarily $x_1t_1x_2t_2$ is a nicely separating, a contradiction to Claim 9.4. Hence there is at most one vertex of H adjacent to $[x_2]$. Thus Q satisfies (b) in H.

(iii) If two vertices r_1 and r_2 of H are adjacent to $[x_1]$, then no vertex is adjacent to $[x_2]$. Indeed suppose for a contradiction that a vertex t is adjacent to $[v_2]$ none of $\{r_1,r_2,t\}$ is in $\{w_1,w_2\}$ by Claim 9.7 and because $dist(C_1,C_2) \geq 2$. Now contracting the path $tx_3x_4z_2$ into a vertex w, we obtain a $K_{3,3}$ with parts $\{r_1,r_2,w\}$ and $\{x_0,x_1,x_2\}$. This contradicts the planarity of G.

Symmetrically, if two vertices of H are adjacent to $[x_3]$, then no vertex is adjacent to $[x_2]$. Therefore Q satisfies (c) in H.

It follows that Q is a good path in $H' = (G-z_2) \setminus z_1x_4$. Let ϕ be a safe L-colouring of Q in H' obtained by Lemma 8. Since Q meets the two crossings, $G-Q$ is planar. Furthermore, $G-Q$ can be drawn in the plane such that all vertices on the outer face are those in $N(Q)$. Every vertex of $Z_Q \setminus \{z_1,z_2\}$ is safe in H' and so in G, so $|L_\phi(v)| \geq 3$. In H', z_1 is safe and in G, z_1 has one more neighbour in Q in G than H', namely x_4. Thus in G, $|L_\phi(z_1)| \geq 2$ because z_1 was safe in H'. Since z_2 has at most four neighbours in Q, we have $|L_\phi(z_2)| \geq 1$. Now z_1 is adjacent to z_2, so L_ϕ is a $\{z_1,z_2\}$-suitable assignment for $G-Q$. Hence by Theorem 2, $G-Q$ is L_ϕ-colourable and so G is L-colourable, a contradiction. □

- Assume first that $|N_Q(z)| = 5$. Let $H = G \setminus \{zx_0, zx_4\}$. z is the unique vertex adjacent to x_0 and x_4. Moreover by property P_3 z is the unique vertex adjacent to x_0 and x_3 and the unique one adjacent to x_1 and x_4. Hence Q satisfies (a) in H. Moreover, for $1 \leq i \leq 3$, there is at most one vertex distinct form z adjacent to $[x_i]$ otherwise $G[V(Q) \cup Z_Q]$ would contain a $K_{3,3}$. Hence Q also satisfies (b) and (c) in H. Therefore Q is great in H. By Lemma 7, there exists a safe L-colouring ϕ of Q in H. Thus in G, every vertex in $Z_Q \setminus \{z\}$ satisfies $|L_\phi(v)| \geq 3$ while $|L_\phi(z)| \geq 1$. Hence L_ϕ is suitable for $G-Q$. Therefore, by Theorem 2, $G-Q$ is L_ϕ-colourable and so G is L-colourable, a contradiction.

- Assume now that $|N_Q(z)| \leq 4$.

  Suppose that there are two distinct vertices $z_1, z_2 \in Z_Q$ with z_1 adjacent to x_0 and x_3 and z_2 adjacent to x_1 and x_4. Let R_1 be the region bounded by the cycle $x_0x_1x_2x_3z_1$ not containing z_2 and R_2 be the region bounded by the cycle $x_1x_2x_3x_4z_2$ not containing z_1 (see Figure 4). Now, note that any vertex adjacent to both x_0 and x_4 is not in $R_1 \cup R_2$ and any vertex adjacent to x_2 must be in $R_1 \cup R_2$. Therefore, $z \in \{z_1,z_2\}$. Indeed if this was not true, then by property P_3 z is not adjacent to x_1 nor x_3. Thus z must be adjacent to x_2 as it is in Z_Q. So z is inside $R_1 \cup R_2$, which contradicts the fact that it is adjacent to x_0 and x_4.

  Thus, at most one other vertex z' in $Z_Q \setminus \{z\}$ is adjacent to vertices at distance three in Q. By symmetry, we may assume that z' is adjacent to x_0 and x_3. Hence all vertices in $Z_Q \setminus \{z,z'\}$ are adjacent to some $[x_i]$ for $1 \leq i \leq 3$. Similarly to (ii) and (iii) in Claim 9.12, one shows that Q also satisfies (a) and (b) in $(G-z) \setminus z'x_0$. Hence Q is a good path in $G-z$. Then Q is magic, a contradiction to Claim 9.5.

□

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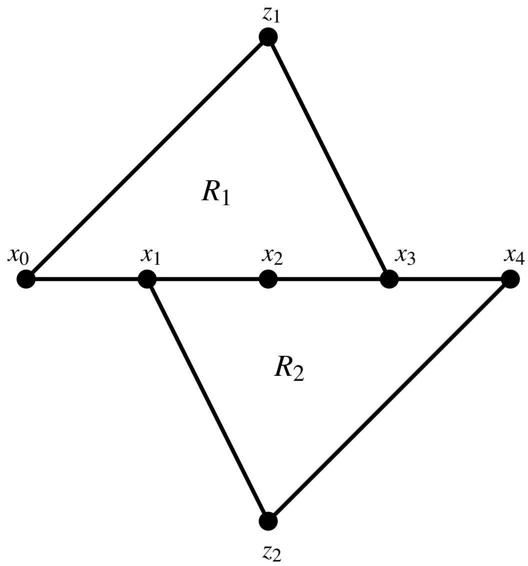

Figure 4: Regions R_1 and R_2.

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